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On homogeneous vector bundles

Indranil Biswas *, S. Subramanian

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India

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Abstract

Let G be a simply connected linear algebraic group, defined over the field of complex numbers, whose Lie algebra is simple. Let P be a proper parabolic subgroup of G. Let E be a holomorphic vector bundle over G/P such that E admits a homogeneous structure. Assume that E is not stable. Then E admits a homogeneous structure with the following property: There is a nonzero subbundle $F \subsetneq E$ left invariant by the action of G such that $\text{degree}(F)/\text{rank}(F) \geqslant \text{degree}(E)/\text{rank}(E)$.

MSC: 14L30; 14M17

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1. Introduction

Let G be a connected linear algebraic group defined over the field of complex numbers such that the Lie algebra is simple. Let P be a proper parabolic subgroup of G. A homogeneous vector bundle over G/P is a holomorphic vector bundle equipped with a holomorphic lift of the right translation action of G on G/P (see Section 2 for the details). Any homogeneous vector bundle given by an irreducible representation of P is known to be stable [3,5].

We prove the following theorem (see Theorem 4.1):

Theorem 1.1. *Assume that G is simply connected. Let E be a holomorphic vector bundle over G/P such that E admits a homogeneous structure. Assume that E is not stable. Then E admits*

* Corresponding author.
E-mail addresses: indranil@math.tifr.res.in (I. Biswas), subramnn@math.tifr.res.in (S. Subramanian).

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a homogeneous structure with the following property: *There is a nonzero subbundle $F \subsetneq E$ left invariant by the action of G such that*

$$\mu(F) := \operatorname{degree}(F)/\operatorname{rank}(F) \geqslant \operatorname{degree}(E)/\operatorname{rank}(E) =: \mu(E).$$

We also give an example showing that the condition in Theorem 1.1 that G is simply connected is essential (see Remark 4.2).

Let E be a $\mathrm{SL}(n+1,\mathbb{C})$-homogeneous vector bundle over $\mathbb{CP}^n$. In Corollary 2.11 of [4] it is asserted that if E is not stable, then E admits a homogeneous nonzero proper subbundle F with $\mu(F) \geqslant \mu(E)$, i.e., the invariant subbundle F violates the stability condition for E. We give an example contradicting this assertion (see Section 3). In view of Theorem 1.1, given a vector bundle E over $\mathbb{CP}^n$ admitting a $\mathrm{SL}(n+1,\mathbb{C})$-homogeneous structure, the assertion in [4, Corollary 2.11] holds for some $\mathrm{SL}(n+1,\mathbb{C})$-homogeneous structure on E.

2. Homogeneous vector bundles

Let G be a connected linear algebraic group defined over $\mathbb{C}$ such that the Lie algebra $\mathfrak{g}$ of G is simple. Let $P \subset G$ be a proper parabolic subgroup. So G/P is a smooth projective variety of positive dimension. Let

$$\rho : G \longrightarrow \operatorname{Aut}(G/P) \tag{2.1}$$

be the homomorphism that sends any $g \in G$ to the automorphism of G/P defined by $x \longmapsto gx$.

A *homogeneous vector bundle* over G/P is a holomorphic vector bundle E equipped with a holomorphic action of the group G lifting the action of G on G/P defined by ρ. In other words, for each point $g \in G$ we are given a holomorphic isomorphism of vector bundles $\delta(g) : \rho(g^{-1})^*E \longrightarrow E$ such that

- $\delta(e) = \mathrm{Id}_E$, and
- $\delta(g_2 g_1) = \delta(g_2) \circ \delta(g_1)$ for all $g_1, g_2 \in G$.

Lemma 2.1. *Let E be a holomorphic vector bundle over G/P such that $\rho(g)^*E$ is holomorphically isomorphic to E for all $g \in G$, where ρ is the homomorphism in* (2.1). *If G is simply connected, then there is an action of G on E making it a homogeneous vector bundle.*

Proof. Consider all pairs of the form (g,h), where $g \in G$ and $h : \rho(g^{-1})^*E \longrightarrow E$ a holomorphic isomorphism of vector bundles. It is easy to see that they form a complex Lie group; this group will be denoted by $\widetilde{G}$. Therefore, we have an exact sequence of groups

$$e \longrightarrow \operatorname{Aut}(E) \longrightarrow \widetilde{G} \longrightarrow G \longrightarrow e, \tag{2.2}$$

where $\operatorname{Aut}(E)$ is the group of all holomorphic automorphisms of the vector bundle E. We note that the group $\widetilde{G}$ has a tautological action on the total space E which is defined by $(g,h) \cdot v = h(v)$. Note that there is a tautological map from the total space of a pullback of any bundle W to the total space of W, in particular, we have a tautological map from the total space of $\rho(g^{-1})^*E$ to that of E; since the action of g^{-1} on G/P gives an automorphism of G/P, this map from the total space of $\rho(g^{-1})^*E$ to that of E is invertible. Therefore, the map $h : \rho(g^{-1})^*E \longrightarrow E$ can be considered as an automorphism of the total space of E.

Let

$$0 \longrightarrow H^0\big(G/P, \operatorname{End}(E)\big) \longrightarrow \widetilde{\mathfrak{g}} \xrightarrow{p} \mathfrak{g} \longrightarrow 0 \tag{2.3}$$

be the exact sequence of Lie algebras obtained from (2.2), where $\widetilde{\mathfrak{g}}$ and $\mathfrak{g}$ are the Lie algebras of $\widetilde{G}$ and G respectively.

Since $\mathfrak{g}$ is simple, the exact sequence in (2.3) admits a right splitting [1, p. 91, Corollaire 3]; this means that there is a homomorphism of Lie algebras

$$f : \mathfrak{g} \longrightarrow \widetilde{\mathfrak{g}} \tag{2.4}$$

such that $p \circ f = \mathrm{Id}_{\mathfrak{g}}$, where p is the homomorphism in (2.3).

Assume that G is simply connected. Therefore, the homomorphism f in (2.4) gives a unique homomorphism of complex Lie groups

$$\phi : G \longrightarrow \widetilde{G}$$

whose differential is f. The group G now acts on the total space of E through the homomorphism ϕ (recall that $\widetilde{G}$ acts on E). It is easy to see that this action makes E a homogeneous vector bundle over G/P. This completes the proof of the lemma. □

Remark 2.2. If G is not simply connected, then the vector bundle E in Lemma 2.1 need not be homogeneous. For example, take $G = \mathrm{PSL}(n+1, \mathbb{C})$, and take $G/P = \mathbb{CP}^n$. For any integer m, the line bundle $\mathcal{O}_{\mathbb{CP}^n}(m)$ over $\mathbb{CP}^n$ has the property that $\rho(g)^*\mathcal{O}_{\mathbb{CP}^n}(m)$ is holomorphically isomorphic to $\mathcal{O}_{\mathbb{CP}^n}(m)$ for all $g \in G$. However, $\mathcal{O}_{\mathbb{CP}^n}(m)$ is homogeneous (with respect to the action of $\mathrm{PSL}(n+1, \mathbb{C})$) if and only if m is an integral multiple of $n+1$.

Fix a cohomology class $\theta \in H^2(G/P, \mathbb{Z})$. Define *degree* of a coherent sheaf F on G/P to be

$$\mathrm{degree}(F) := \big(\theta^{\dim G/P - 1} \cup c_1(F)\big) \cap [G/P] \in \mathbb{Z}.$$

The quotient $\mathrm{degree}(F)/\mathrm{rank}(F)$ is called the *slope* of F, and it is denoted by $\mu(F)$.

A torsionfree coherent sheaf E over G/P is called *stable* (respectively, *semistable*) (with respect to θ) if $\mu(F) < \mu(E)$ (respectively, $\mu(F) \leqslant \mu(E)$) for all coherent subsheaves F of E with $0 < \mathrm{rank}(F) < \mathrm{rank}(E)$. A semistable sheaf is called *polystable* if it is a direct sum of stable sheaves.

3. A lemma and an example

We start with the following lemma.

Lemma 3.1. *Let E be a homogeneous vector bundle over G/P.*

(1) *If $\mu(F) \leqslant \mu(E)$ for all nonzero coherent subsheaves $F \subset E$ left invariant by the action of G on E, then E is semistable.*
(2) *If $\mu(F) < \mu(E)$ for all nonzero coherent proper subsheaves $F \subsetneq E$ left invariant by the action of G on E, then E is polystable.*

Proof. If E is not semistable, let $F_1 \subset E$ be the first term of the Harder–Narasimhan filtration of E [2, p. 16, Theorem 1.3.4]. Note that as G is connected, the cohomology class θ is left invariant by the action of G on G/P. From the uniqueness of the Harder–Narasimhan filtration it follows that F_1 is left invariant by the action of G on E. This proves the first part of the lemma.

Assume that $\mu(F) < \mu(E)$ for all coherent subsheaves $0 \neq F \subsetneq E$ left invariant by the action of G on E. The first part of the lemma implies that E is semistable. Let $F \subset E$ be the socle of the semistable vector bundle [2, Lemma 1.5.5, p. 23]. We recall that the socle is the unique maximal polystable subsheaf of E with slope $\mu(E)$. From the uniqueness of the socle it follows that F is left invariant by the action of G on E. Therefore, $F = E$. This completes the proof of the lemma. □

We will give an example to show that the polystable vector bundle E in the second part of Lemma 3.1 need not be stable.

Let V be an irreducible complex left G-module of dimension at least two. Let E_V be the trivial vector bundle $(G/P) \times V$ over G/P. The group G acts on E_V as follows. The action of each point $g \in G$ sends any point $(z, v) \in (G/P) \times V$ to $(gz, g \cdot v)$. This action makes E_V a homogeneous vector bundle over G/P.

It is easy to see that all the G-invariant subbundles of E_V are given by the G-invariant subspaces of V. Since V is an irreducible G-module, the vector bundle E_V does not have any nonzero G-invariant proper subbundle. As $\dim V \geqslant 2$, the vector bundle E_V is not stable. Therefore, E_V is an example of a polystable homogeneous vector bundle E satisfying the condition in the second part of Lemma 3.1 which is not stable.

4. Unstable homogeneous bundles

We will now prove the main result.

Theorem 4.1. *Assume that G is simply connected. Let E be a holomorphic vector bundle over G/P such that E admits a homogeneous structure. Assume that E is not stable. Then E admits a homogeneous structure ρ_0 with the following property: there is a nonzero proper holomorphic subbundle $F \subsetneq E$ left invariant by the action of G (with respect to ρ_0) such that $\mu(F) \geqslant \mu(E)$.*

Proof. Fix an action ρ' of G on E that makes E a homogeneous vector bundle. If E is not polystable, then there is a G-invariant subsheaf F satisfying the above condition (see the second part of Lemma 3.1). Therefore, we may assume that E is polystable.

Assume that E is polystable. Let

$$E = \bigoplus_{i=1}^{\ell} E_i \tag{4.1}$$

be a decomposition of E into a direct sum of stable vector bundles. So, $\mu(E) = \mu(E_i)$ for all $i \in [1, \ell]$. As E is not stable, we have $\ell > 1$. Set

$$A = \{i \in [1, \ell] \mid E_i \cong E_1\} \subset [1, \ell], \tag{4.2}$$

where E_i are as in (4.1). Set

$$E' := \bigoplus_{i \in A} E_i \subset E, \tag{4.3}$$

where A is defined in (4.2).

If F_1 and F_2 are two stable vector bundles over G/P, with $\mu(F_1) = \mu(F_2)$, which are not isomorphic, then

$$H^0(G/P, F_1^* \otimes F_2) = 0.$$

Also, any nonzero holomorphic endomorphism of a stable vector bundle is an isomorphism. Using these it follows that any holomorphic automorphism of the vector bundle E preserves the subbundle E' constructed in (4.3). Consequently, E' is left invariant by the action of G on E (given by ρ'). Therefore, if E' is a proper subbundle of E, then we can take the pair (ρ_0, F) in the statement of the theorem to be (ρ', E'). Hence we may assume that $E = E'$.

Assume that $E = E'$. Therefore, $E = E' \cong E_1 \otimes_{\mathbb{C}} \mathbb{C}^\ell$. Identify E with $E_1 \otimes_{\mathbb{C}} \mathbb{C}^\ell$ by fixing an isomorphism. Therefore, G acts on $E_1 \otimes_{\mathbb{C}} \mathbb{C}^\ell$ through ρ' and the isomorphism making $E_1 \otimes_{\mathbb{C}} \mathbb{C}^\ell$ a homogeneous vector bundle. This action of G on $E_1 \otimes_{\mathbb{C}} \mathbb{C}^\ell$ will be denoted by ρ''.

We will show that the vector bundle E_1 admits a homogeneous structure. For any $g \in G$, the vector bundle $\rho(g)^* E_1$ will be denoted by E_1^g, where ρ is the homomorphism in (2.1). Therefore, E_1^g is stable with $\mu(E_1) = \mu(E_1^g)$. Hence any homomorphism between E_1 and E_1^g is either the zero homomorphism or it is an isomorphism. We have $\rho(g)^*(E_1 \otimes_{\mathbb{C}} \mathbb{C}^\ell) = E_1^g \otimes_{\mathbb{C}} \mathbb{C}^\ell$. We note that $\rho''(g^{-1})$ is an isomorphism of $E_1 \otimes_{\mathbb{C}} \mathbb{C}^\ell$ with $E_1^g \otimes_{\mathbb{C}} \mathbb{C}^\ell$ (the action ρ'' of G is defined above). Since any nonzero homomorphism between E_1 and E_1^g is an isomorphism, we conclude that the vector bundle E_1 is holomorphically isomorphic to E_1^g. Now from Lemma 2.1 it follows that E_1 admits a homogeneous structure.

Let $\widetilde{\rho}$ be an action of G on E_1 that makes E_1 a homogeneous vector bundle.

Let ρ_t denote the trivial action of G on $\mathbb{C}^\ell$. Therefore, $\widetilde{\rho} \otimes \rho_t$ is an action of G on $E_1 \otimes_{\mathbb{C}} \mathbb{C}^\ell = E$ making $E_1 \otimes_{\mathbb{C}} \mathbb{C}^\ell$ a homogeneous vector bundle.

Fix a line $\xi \subset \mathbb{C}^\ell$; recall that $\ell > 1$. The proper subbundle

$$E_1 \cong E_1 \otimes_{\mathbb{C}} \xi \subset E_1 \otimes_{\mathbb{C}} \mathbb{C}^\ell = E$$

is preserved by the action $\widetilde{\rho} \otimes \rho_t$, and $\mu(E_1 \otimes_{\mathbb{C}} \xi) = \mu(E)$. Therefore, we can take the pair (ρ_0, F) in the statement of the theorem to be $(\widetilde{\rho} \otimes \rho_t, E_1 \otimes_{\mathbb{C}} \xi)$. This completes the proof of the theorem. □

Remark 4.2. The assumption in Theorem 4.1 that G is simply connected is essential. To see this, let V be any complex vector space of dimension d, with $d \geqslant 2$. Let $\mathbb{P}(V)$ denote the projective space of hyperplanes in V. Fix a hyperplane $S_0 \subset V$. Set $G = \mathrm{PGL}(V)$, and set P to be the parabolic subgroup of $\mathrm{PGL}(V)$ that fixes S_0 for the natural action of $\mathrm{PGL}(V)$ on $\mathbb{P}(V)$. Therefore, $G/P = \mathbb{P}(V)$. Let E be the vector bundle $\mathcal{O}_{\mathbb{P}(V)}(1) \otimes_{\mathbb{C}} V^*$ over $\mathbb{P}(V)$. The group $\mathrm{PGL}(V)$ has a natural action on E. To see this action, note that for the natural action of $\mathrm{GL}(V)$ on $\mathcal{O}_{\mathbb{P}(V)}(1) \otimes_{\mathbb{C}} V^*$, the center of $\mathrm{GL}(V)$ acts trivially inducing an action of $\mathrm{PGL}(V)$. It can be shown that there is no other action of $\mathrm{PGL}(V)$ on E making it a homogeneous vector bundle. Indeed, any two such actions on E differ by a homomorphism from $\mathrm{PGL}(V)$ to $\mathrm{Aut}(E) = \mathrm{GL}(V)$, and there is no nontrivial homomorphism from $\mathrm{PGL}(V)$ to $\mathrm{GL}(V)$. It is easy to check that the natural action of $\mathrm{PGL}(V)$ on $E = \mathcal{O}_{\mathbb{P}(V)}(1) \otimes_{\mathbb{C}} V^*$ has the following property: there is no proper nonzero holomorphic subbundle F of E with $\mu(F) = \mu(E)$ that is preserved by the action of $\mathrm{PGL}(V)$ on E. In fact, the only proper nonzero subbundle of E that is preserved by $\mathrm{PGL}(V)$ is a line subbundle of degree zero. These can be proved by considering the action of the isotropy subgroup of any point $z \in G/P$ on the fiber E_z. For the action of P on $\mathbb{P}(V)$, the only proper linear subspace that is fixed is the point given by S_0. Therefore, the condition in Theorem 4.1 that G is simply connected is essential.

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